1.9. Brackets

Evaluating a mathematical expression follows certain rules regarding the orders of operations. Multiplication and division have priority over addition and subtraction. If the order of operations is not followed correctly, the result will most likely be incorrect.

The positive of a negative number is negative, but the negative of a negative number is positive. Remember that for any number a the following identities are always valid:

$$
\begin{aligned}
&+(+a) = +(a) = +a = a \\
&+(-a) = -a \\
&-(+a) = -(a) = -a \\
&-(-a) = +a = a
\end{aligned}
$$

Note: When confused with letters inside equations, try to replace the letters with small numbers to try to figure out what's going on. Make sure to replace each occurrence of the letter with the same number. For example, when replacing "a" with 3, replace every "a" with 3.

Practice 1: Remove brackets.

a) $+(+2) = 2$

b) $+(-2) = -2$

c) $-(-2) = 2$

d) $-(+2) = -2$

e) $-(-x) = x$

f) $+(-x) = -x$

g) $+(+x) = x$

h) $-(+x) = -x$

i) $-(+2.3) = -2.3$

j) $+(-0.4) = -0.4$

k) $-(-0.1) = 0.1$

l) $+(+1.5) = 1.5$

If the plus sign $(+)$ is in front of the brackets, the brackets can be erased, if the minus sign $(-)$ is in front of the brackets, all terms inside the brackets are changing signs.

Example 1: Get rid of the brackets and solve

$$(+3) - (+4) + (-2) - (-6) + \left(-\frac{1}{2}\right) - \left(-\frac{3}{2}\right) =$$

Solution: *If the plus sign $(+)$ is in front of the brackets, the brackets can be erased, if the minus sign $(-)$ is in front of the brackets, all terms inside the brackets are changing signs.*

$$= \ +3 - 4 - 2 + 6 - \frac{1}{2} + \frac{3}{2} \ = \ 3 + \frac{2}{2} \ = \ 3 + 1 \ = \ 4$$

Note: In math, brackets always come in pairs. A singular form "bracket" is used for a left/opening bracket '(' and for a right/closing bracket ')'.

Practice 2: Get rid of the brackets and solve

a) $2-(-4)+(-5) = 2+4-5 = 6-5 = 1$

b) $-1-(-1)+(-1) = -1+1-1 = 0-1 = -1$

c) $0.05+(-0.1)-(-0.03) = 0.05-0.1+0.03 = -0.05+0.03 = -0.02$

d) $1\cdot(-4)\cdot(-5) = (-4)\cdot(-5) = +20$

e) $-0.1\cdot(-0.1)\cdot(0.01) = 0.01\cdot 0.01 = 0.0001 = 10^{-4}$

f) $0.1\div(-0.01) = -\dfrac{0.1}{0.01} = -\dfrac{10}{1} = -10$

g) $\dfrac{1}{2}-\left(-\dfrac{1}{3}\right)+\left(-\dfrac{3}{4}\right) = \dfrac{1}{2}+\dfrac{1}{3}-\dfrac{3}{4} = \dfrac{1\cdot 2\cdot 3+1\cdot 4-3\cdot 3}{3\cdot 4} = \dfrac{6+4-9}{12} = \dfrac{1}{12}$

h) $100-(-1000)+(-10)-(-1) = 100+1000-10+1 = 1100-9 = 1091$

i) $0.5+\left(-\dfrac{1}{3}\right)-\left(-\dfrac{1}{2}\right) = \dfrac{1}{2}-\dfrac{1}{3}+\dfrac{1}{2} = \dfrac{1\cdot 3-1\cdot 2+1\cdot 3}{2\cdot 3} = \dfrac{3-2+3}{6} = \dfrac{4}{2\cdot 3} = \dfrac{2}{3}$

Multiplication is Performed Prior to Addition

Example 2: The order of operations is very important!

$$2+2\times 2 \;=\; 2+4 \;=\; 6$$ multiplication is performed prior to addition

We can prove the above calculation by using the commutative property of addition. Just switch the terms with respect to addition

$$2\times 2+2 \;=\; 4+2 \;=\; 6$$

If we attempt to evaluate $2+2\times 2$ by working our way from left to right without following the order of operations, we would first add $2+2$ and multiply the result by 4.

$$\underbrace{2+2}_{first}\times 2 \;\neq\; 4\times 2 \;=\; 8$$ Incorrect!

In mathematics, brackets indicate a grouping that affects the order of operations. Brackets override normal precedence by grouping terms and operators together. An opening brackets marks the beginning and a closing brackets marks the end of a group. Anything inside the brackets behaves as a single entity, thus affecting the order of operations.

Example 3:

$$(2+2)\times 2 \;=\; (4)\times 2 \;=\; 4\times 2 \;=\; 8$$

Notice the difference the brackets introduced compared to the previous example.

Types of Brackets

Different types of brackets have multiple uses in mathematics.

Brackets used for grouping in expressions are:

- (...) Round brackets or parentheses. In addition to usage in grouping, they are also used to separate arguments of functions such as $f(x)$, where x is a parameter of a function f; to separate pairs of coordinates such as A(1, 2); and to denote open intervals (1, 5).
- [...] Square brackets or simply brackets (US). Also used to denote a closed end of interval, such as [1, 5], where the interval includes number 1, but not number 5.
- {...} Curly brackets or braces. Also used to denote elements of sets such as the first five natural numbers {1, 2, 3, 4, 5}.

Nesting

More complicated expressions use brackets inside brackets, or **nested** brackets. The common practice is to use the following order when having three nested structures:

$$\Big\{ [(...)] \Big\}$$

If there are more than three levels of nesting, the symbols are being repeated.

Remember: Make sure that inner bracket is closed before closing outer bracket.

Example 4: The nested brackets must be properly closed.

$[2 + (3 -] + 1) - 1 =$	Incorrect
$[2 + (3 - 1)] - 1 =$	Correct

The rule of dealing with nested brackets is to first process the most inner group and work your way toward the most outer group.

Practice 3: Evaluate.

a) $5 - [3 + (2 - 1)] = 5 - [3 + (1)] = 5 - [3 + 1] = 5 - [4] = 5 - 4 = 1$

b) $2 \cdot (1 + 2) = 2 \cdot (3) = 2 \cdot 3 = 6$

c) $-[2 - (1 - 5)] + 1 = -[2 - (-4)] + 1 = -[2 + 4] + 1 = -[6] + 1 = -6 + 1 = -5$

Example 5:

$$\begin{aligned}\frac{1}{2}\cdot\left[2-\frac{1}{2}\cdot(4+8\div 2)\right] &= && \text{first process division inside ( )}\\ &=\frac{1}{2}\cdot\left[2-\frac{1}{2}\cdot(4+4)\right] && \text{complete calculations of ( )}\\ &=\frac{1}{2}\cdot\left[2-\frac{1}{2}\cdot 8\right] && \text{perform multiplication inside [ ]}\\ &=\frac{1}{2}\cdot[2-4] && \text{complete [ ]}\\ &=\frac{1}{2}\cdot[-2] && \text{finish computation}\\ &=-1\end{aligned}$$

Practice 4: Evaluate:

a) $2-[2\cdot(3+1)] = 2-[2\cdot 4] = 2-8 = -6$

b) $\frac{1}{3}\cdot\left[\frac{1}{2}-\left(-\frac{1}{2}+1\right)\right] = \frac{1}{3}\cdot\left[\frac{1}{2}-\frac{1}{2}\right] = \frac{1}{3}\cdot 0 = 0$

c) $-\left[1-\left(\frac{1}{4}+\frac{1}{2}\right)\right] = -\left[1-\frac{1+1\cdot 2}{4}\right] = -\left[\frac{4}{4}-\frac{3}{4}\right] = -\frac{1}{4}$

d) $a\cdot(0.5-(0.1-(-0.1))) = a(0.5-(0.1+0.1)) = a(0.5-0.2) = a\cdot 0.3 = 0.3a$

Expanding Brackets

A procedure of removing brackets by multiplying everything inside the bracket by the number (or letter) outside the bracket is called **expanding** or **removing** brackets. A bracket multiplied by some value can be removed by multiplying each of the terms inside the bracket by that value:

$$(x+y)\cdot a = a\cdot(x+y) = ax+ay$$

$$(x-y)\cdot a = a\cdot(x-y) = ax-ay$$

Example 6: Remove brackets.

$$\frac{1}{2}\cdot\left(x-\frac{1}{3}\right) = \frac{1}{2}x+\frac{1}{2}\cdot\left(-\frac{1}{3}\right) = \frac{1}{2}x-\frac{1}{6}$$

Practice 5: Simplify by getting rid of the brackets.

a) $2\cdot(1+x) = 2+2x$

b) $0.5\cdot(2+x) = 1+0.5x$

c) $(3+2)\cdot 2 = 6+4 = 10$

d) $4\cdot(x+2y) = 4x+8y$

A minus sign outside brackets changes all the signs inside the brackets when the brackets are removed.

Example 7: Remove brackets.

$$-(2 + x + 4x^2) = -2 - x - 4x^2$$

Note: A minus sign in front of brackets is considered as multiplication by -1*.*

Practice 6: Simplify by getting rid of the brackets.

a) $-(1 + x) = -1 - x$

b) $-(1 - x) = -1 + x$

c) $-(-3 + 2) = 3 - 2 = 1$

d) $-(x + 2y) = -x - 2y$

Note: A value outside a bracket tells us that everything inside that bracket is multiplied by that value. For example: $2(a + b) = 2 \cdot (a + b)$.

Practice 7: Simplify by getting rid of the brackets.

a) $-2 \cdot (a - 1) = -2a + 2$

b) $3 \cdot \left(\frac{1}{3} + x\right) = 1 + 3x$

c) $\frac{1}{2}(x - 4) = \frac{x}{2} - \frac{4}{2} = \frac{x}{2} - 2$

d) $0.5a \cdot \left(\frac{1}{2} - a\right) = \frac{1}{2}a \cdot \left(\frac{1}{2} - a\right) = \frac{1}{2}a \cdot \frac{1}{2} - \frac{1}{2}a \cdot a = \frac{a}{4} - \frac{1}{2}a^2$

Challenge 1: Simplify by getting rid of the brackets.

a) $3 \cdot [1 + 2 \cdot (a - 1)] = 3 \cdot [1 + 2a - 2] = 3 \cdot [2a - 1] = 6a - 3$

b) $2 \cdot [x - 2 \cdot (1 - 2x + 1)] = 2 \cdot [x - 2(2 - 2x)] = 2 \cdot [x - 4 - 4x] = 2 \cdot [-3x - 4] = -6x - 8$

Multiplying Brackets

When multiplying two brackets, multiply each term from the first bracket with each term of the second bracket:

$$(a + b) \cdot (d + e) = ad + ae + bd + be$$

Example 8:

$$(a + 4)(2 + b) = a \cdot 2 + a \cdot b + 4 \cdot 2 + 4 \cdot b = 2a + ab + 8 + 4b$$

Subtraction is the same as addition of a negative value.

Example 9:

$$(a+4)(2-b) = a\cdot 2 + a\cdot(-b) + 4\cdot 2 + 4\cdot(-b) = 2a - ab + 8 - 4b$$

Practice 8: Expend by getting rid of the brackets.

a) $(2x+y)(x+y) = 2x\cdot x + 2x\cdot y + y\cdot x + y\cdot y = 2x^2 + 2xy + xy + y^2 = 2x^2 + 3xy + y^2$

b) $(x+y)^2 = (x+y)(x+y) = x\cdot x + x\cdot y + y\cdot x + y\cdot y = x^2 + 2xy + y^2$

c) $(x-y)^2 = (x-y)(x-y) = x\cdot x - x\cdot y - y\cdot x - y\cdot(-y) = x^2 - 2xy + y^2$

d) $\left(\frac{1}{2}+x\right)^2 = \left(\frac{1}{2}+x\right)\left(\frac{1}{2}+x\right) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot x + x\cdot\frac{1}{2} + x\cdot x = \frac{1}{4} + 2\frac{x}{2} + x^2 = \frac{1}{4} + x + x^2$

Challenge 2: Prove identity: $(a-b)(a+b) = a^2 - b^2$.

$$(a-b)(a+b) = a\cdot a + a\cdot b - b\cdot a - b\cdot b = a^2 + ab - ab - b^2 = a^2 - b^2$$